Math 259A Lecture 11 Notes

Daniel Raban

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1 Multiplication Operators on L^2

1.1 Multiplication operators on L^2

Let $(X, \mathcal{F}, \mu)$ be a probability space (we usually assume $\mathcal{F}$ to be countably generated). Then X is measurably isomorphic to $[0,1] \oplus \{1,2,\dots\}$, where $[0,1]$ has (scaled) Lebesgue measure and $\{1,2,\dots\}$ has some scaled counting measure on a subset. If X is a compact metric space, we usually take $\mathcal{F}$ to be the σ-algebra of Borel sets. We choose μ to be regular and assume $\operatorname{supp}(x) = X$. In particular, we may take X to be the spectrum of an operator $x \in \mathcal{B}(H)$.

Consider $L^\infty(X,\mu)$ with $\|\cdot\|_\infty$. If we define $f^*(t) = \overline{f(t)}$, then we get a C^*-algebra structure. If $f \in L^\infty$, we get a multiplication operator on $L^2(\mu)$ given by $M_f(g) = fg$. Moreover, $\|fg\|_2 \leq \|f\|_\infty \|g\|_2$, so $M_f \in \mathcal{B}(L^2)$ with $\|M_f\| \leq \|f\|_\infty$.

Proposition 1.1. $\|M_f\| {=} \|f\|_\infty$.

Proof. Let $X_m = \{t \in X : |f(t) \geq \|f\|_\infty - 1/n\}$. Then $\mu(X_m) > 0$. Then

$$\|M_f(\mathbb{1}_{X_m})\|_2 = \int_{X_n} |f|^2 \, d\mu \geq \left(\left(\|f\|_\infty - \frac{1}{m} \right)^2 \mu(X_m) \right)^{1/2} = (\|f\|_\infty - \frac{1}{m}) \|\mathbb{1}_{X_m}\|_2.$$

So if $\xi_n = \|\mathbb{1}_{X_m}\|_2^{-1} \mathbb{1}_{X_m}$, then we get $\|M_f \xi_n\|_2 \geq \|f\|_\infty - 1/m$. ☐

Corollary 1.1. *$f \mapsto M_f$ is an isometric *-algebra morphism from L^∞ into $\mathcal{B}(H)$.*

Proof. We have that $f \mapsto M_f$ is a *-algebra morphism, and $M_{\overline{f}} = (M_f)^*$. ☐

Theorem 1.1. *$\mathcal{A} := \{M_f : f \in L^\infty\} \subseteq \mathcal{B}(L^2)$ is a von Neumann algebra (i.e. it is WO-closed). Moreover, $\mathcal{A}' = \mathcal{A}$ (so $\mathcal{A}$ is maximal abelian in $\mathcal{B}(L^2)$).*

Proof. By von Neumann's bicommutant theorem, we need only show that $\mathcal{A}' = \mathcal{A}$. Let $T \in \mathcal{B}(L^2)$, and suppose that $TM_f = M_f T$ for all $f \in L^\infty$. Then let $\varphi := T(1) \in L^2$.

Define $M_\varphi : L^2 \to L^1$ by $M_\varphi(\psi) = \varphi\psi$ (the image is in L^1 by Cauchy-Schwarz). Then $\|M_\varphi\|_{\mathcal{B}(L^2,L^1)} \leq \|\varphi\|_2$ by Cauchy-Schwarz. Both T, M_φ are continuous from $L^2 \to L^1$, and they coincide on $L^\infty \subseteq L^2$ because if $f \in L^\infty$,

$$T(f) = TM_f(1) = M_f T(1) = M_f(\varphi) = f\varphi = M_\varphi(f).$$

Since L^∞ is dense in L^2, $T = M_\varphi$ as operators in $\mathcal{B}(L^2, L^1)$. So $M_\varphi(L^2) \subseteq L^2$.

Why is $\varphi \in L^\infty$? Assume that $\varphi \notin \ell^\infty$. Let $X_n = \{t \in X : |\varphi(t)| \geq n\}$, and let $\xi_n = \mu(X_n)^{-1/2} \mathbb{1}_{X_n}$. Then $\|M_\varphi(\xi_n)\|_2 \geq n$. Letting $n \to \infty$ yields a contradiction. □

1.2 Sups of dominating sequences of operators

Lemma 1.1. *Let $x = x^* \in \mathcal{B}(H)_h$ and let $f_n, g_m \geq 0$ be increasing sequences of continuous functions on $\operatorname{Spec}(x)$ that are both uniformly bounded. If $\sup_n f_n(t) \leq \sup_n g_n(t)$ for all $t \in \operatorname{Spec}(x)$, then $\sup_n f_n(x) \leq \sup_n g_n(x)$.*

Let $e_t := \mathbb{1}_{(t,\infty)}(x)$. We can then build all bounded measurable functions using these, and this will give us a functional calculus for all Borel measurable functions.